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On the price of anarchy for non-atomic congestion games under asymmetric cost maps and elastic demands[☆]

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ABSTRACT

We derive several bounds for the price of anarchy of the noncooperative congestion games with elastic demands and asymmetric linear or nonlinear cost functions. The bounds established depend on a constant from the cost functions as well as the ratio between user benefit and social surplus at Nash equilibrium. The results can be viewed a generalization of that of Chau and Sim [C.K. Chau, K.M. Sim, The price of anarchy for non-atomic congestion games with symmetric cost maps and elastic demands, Operations Research Letters 31 (2003) 327–334] for the symmetric case, or a generalization of Perakis [G. Perakis, The price of anarchy when costs are nonseparable and asymmetric, Lecture Notes in Computer Science 3064 (2004) 46–58] to the elastic demand.

1. Introduction

In noncooperative games, agents perform selfishly, i.e., each agent selects her strategy trying to minimize her own cost. When no agent can improve her cost by unilaterally changing her strategy, the system arrives at the Nash equilibrium, or Wardropian User Equilibrium (UE) in the transportation literature. On the other hand, the system manager hopes that the overall cost in the system is minimized, which corresponds to Wardropian System Optimum (SO) in the transportation literature. Generally, Nash equilibria do not minimize the overall cost in the system. In real urban traffic systems, observed flows are likely to be closer to UE than SO, i.e., there is an efficiency loss in the system.

To characterize the efficiency loss in the system with fixed demand, Koutsoupias and Papadimitriou [3] introduced the notion "the price of anarchy", also known as "coordination ratio", which is the ratio between the system cost at the worst Nash equilibrium and the optimum system cost. Their result was then improved by Czumaj and Vöcking [4]. Subsequently, Roughgarden and Tardos [5], Roughgarden [6], and Correa et al. [7,8] applied this idea to the classical network equilibrium problem in transportation with link cost functions that are separable of arc flows. Chau and Sim [1] extended Roughgarden and Tardos's results to nonseparable but symmetric cost functions. Then Perakis [2] considered the case that the cost function was asymmetric and proved some tight bounds. Han and Sun [9] then proposed some new bounds for the asymmetric and nonlinear, nonseparable cost functions, which depend on an intrinsic constant of the cost function.

All the results mentioned above considered the fixed demand case, except [1], which also considered the elastic demand case. In [1], Chau and Sim proposed a "weaker" bound with symmetric cost function and elastic demand. The bound is "weaker" in the sense that, unlike its counterpart with fixed demand, the bound depends on not only a constant from the cost function itself, but also on the ratio between "user benefit" and "social surplus" at Nash equilibrium (for the definition

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of these two terms, see Definition 2.1 in the next section). This result is from the fact that in the elastic demand case, one cannot simply seek to minimize the total system cost as the system objective (because the minimized system cost is zero). Thus, the price of anarchy in this case is defined as the ratio between the maximum social surplus and the total surplus at user equilibrium.

In this paper, we consider the price of anarchy for the elastic demand case but with asymmetric cost functions. The bounds derived here also depend on the ratio between user benefit and social surplus at the user equilibrium.

In the next section, we list some useful notations and describe our model under consideration. We derive and prove our bounds on the price of anarchy in Section 3 and give two simple examples to illustrate our results in Section 4. Section 5 consists of some concluding remarks.

2. Preliminaries

Let $\mathcal{I} = \{1, 2, \ldots, I\}$ denote a set of populations where population $i \in \mathcal{I}$ has a mass n_i, where each infinitesimal agent within the population is represented by a point in $[0, n_i]$. The aggregated population masses are a vector $\mathbf{n} \in R^I_+$. $\mathcal{F} = \{1, 2, \ldots, F\}$ denotes a set of shared resources where agents collectively generate a utilization rate v_f for each $f \in \mathcal{F}$. The aggregate utilization rates are a vector $\mathbf{v} \in R^F_+$. $\mathcal{S} = \{1, 2, \ldots, S\}$ denotes a set of strategies available to each $i \in \mathcal{I}$. $\mathbf{A}$ is an $I \times S$ population-strategy incidence matrix where entry $a_{i,s}$ is 1 if strategy s is available to i, 0 otherwise. $\mathbf{Z}$ is an $F \times S$ consumption rate matrix with $z_{f,s} \geq 0$ being the consumption rate when f is invoked by strategy s. Let x_s be the strategy distribution of total population mass choosing strategy $s \in \mathcal{S}$ and the aggregated strategy distributions are a vector $\mathbf{x} \in R^S_+$. Variables $\mathbf{x}$, $\mathbf{v}$, $\mathbf{n}$, satisfy the following constraints:

$$\mathbf{v} = \mathbf{Z}\mathbf{x},$$
$$\mathbf{n} = \mathbf{A}\mathbf{x},$$
$$\mathbf{x} \geq 0.$$

For each $\mathbf{n}$, the feasible set of the aggregated strategy distributions is defined as $X(\mathbf{n}) = \{\mathbf{x} \in R^S_+ \mid \mathbf{n} = \mathbf{A}\mathbf{x},\ \mathbf{x} \geq 0\}$, which is a convex set.

The resource cost function is defined as $\mathbf{c}(\mathbf{v}) : R^F_+ \to R^F_+$ with each $c_f(\mathbf{v})$ representing the cost function of f with respect to the aggregated utilization rates. $\mathbf{c}(\mathbf{v})$ is said to be symmetric if $\partial c_f(\mathbf{v})/\partial v_g = \partial c_g(\mathbf{v})/\partial v_f$, $\forall g, f \in \mathcal{F}$, and if $\mathbf{c}(\mathbf{v}) = (c_f(v_f))^\top_{f\in\mathcal{F}}$, i.e., c_f only depends on the utilization rate on f for all $f \in \mathcal{F}$, $\mathbf{c}(\mathbf{v})$ is said to be separable. The cost function $\mathbf{c}(\mathbf{v})$ is said to be linear if $\mathbf{c}(\mathbf{v}) = \mathbf{M}\mathbf{v} + \boldsymbol{\phi}$, where $\mathbf{M}$ is an $F \times F$ matrix and $\boldsymbol{\phi}$ is a vector with $m_{f,g} \geq 0$, $\phi_g \geq 0$, $\forall f, g \in \mathcal{F}$. Then, a tuple $(\mathcal{I}, \mathcal{F}, \mathbf{A}, \mathbf{Z}, \mathbf{c}, \mathbf{n})$ defines a noncooperative congestion game.

If $\mathbf{n}$ is a fixed vector, then the game is a fixed-demand congestion game; otherwise, it is a elastic-demand game and $\mathbf{n}$ is a demand map. That is, $\mathbf{n}(\mathbf{d}) : R^I_+ \to R^I_+$ is the demand function, where $\mathbf{d} \in R^I_+$ is a vector of minimum private costs experienced by each population. We will assume that $\mathbf{n}(\mathbf{d})$ is separable and strictly decreasing. With this assumption, the inverse demand map $\mathbf{d}(\mathbf{n})$ is well defined.

Let $\mathbf{t}(\mathbf{x}) : R^S_+ \to R^S_+$ be the private cost with respect to strategy distribution $\mathbf{x}$, with each component $t_s(\mathbf{x})$ denoting the private cost experienced by an agent who employs strategy s. Here we adopt the additive cost

$$t_s(\mathbf{x}) = \sum_{f\in\mathcal{F}} z_{f,s} c_f(\mathbf{v}),$$

that is, the cost of a strategy distribution of total population mass choosing the strategy x_s is the sum of the cost on f consisting in s. More concisely,

$$\mathbf{t}(\mathbf{x}) = \mathbf{Z}^\top \mathbf{c}(\mathbf{v}).$$

The system is said to arrive at Nash equilibrium, if there is no agent motivating to unilaterally change her strategy. That is, there is a strategy distribution $\bar{\mathbf{x}}$ such that

$$t_s(\bar{\mathbf{x}}) \begin{cases} = d_i, & \text{if } \bar{x}_s > 0, \\ \geq d_i, & \text{if } \bar{x}_s = 0, \end{cases} \quad \forall s \in \mathcal{S}_i, i \in \mathcal{I}, \tag{1}$$

where d_i is the minimum private cost and the aggregated minimum private costs are a vector $\mathbf{d} \in R^I_+$.

According to [10], the Nash equilibrium (1) can be mathematically formulated as a variational inequality

$$\mathbf{t}(\bar{\mathbf{x}})^\top (\mathbf{x} - \bar{\mathbf{x}}) - \mathbf{d}(\bar{\mathbf{n}})^\top (\mathbf{n} - \bar{\mathbf{n}}) \geq 0, \quad \forall \mathbf{x} \in X(\mathbf{n}),\ \mathbf{n} \geq 0. \tag{2}$$

In the following, we will always assume that the Nash equilibrium exists.

Definition 2.1. Assuming $d_i(0) < \infty$, $\forall i \in \mathcal{I}$, the user benefit is defined as

$$U(\mathbf{x}, \mathbf{n}) = \sum_{i\in\mathcal{I}} \int_0^{n_i} d_i(\omega)\mathrm{d}\omega$$

and the social surplus is defined as

$$E(\mathbf{x},\mathbf{n}) = \sum_{i\in\mathcal{I}} \int_0^{n_i} d_i(\omega)\mathrm{d}\omega - \mathbf{x}^\top \mathbf{t}(\mathbf{x}). \quad \square$$

Hence, the system optimum $(\hat{\mathbf{x}},\hat{\mathbf{n}})$ is defined as

$$(\hat{\mathbf{x}},\hat{\mathbf{n}}) = \underset{x\in X(n),n\geq 0}{\operatorname{argmax}}\ E(\mathbf{x},\mathbf{n}) \tag{3}$$

and the total surplus at Nash equilibrium is $E(\bar{\mathbf{x}},\bar{\mathbf{n}})$. Consequently, the price of anarchy for noncooperative congestion game is defined as the ratio $E(\hat{\mathbf{x}},\hat{\mathbf{n}})/E(\bar{\mathbf{x}},\bar{\mathbf{n}})$.

If the demand function $\mathbf{n}$ is a constant, then the game reduces to a noncooperative congestion game with fixed demand and the Nash equilibrium is characterized as

$$\mathbf{t}(\bar{\mathbf{x}})^\top(\mathbf{x}-\bar{\mathbf{x}}) \geq 0, \quad \forall \mathbf{x}\in X,$$

where $X = \{x \in R^S_+ \mid \mathbf{Ax} = \mathbf{n},\ \mathbf{x}\geq 0\}$. The system optimum $\hat{\mathbf{x}}$ is defined as

$$\hat{\mathbf{x}} = \underset{x\in X}{\operatorname{argmin}}\ \mathbf{x}^\top\mathbf{t}(\mathbf{x})$$

and the total system cost at Nash equilibrium is $\bar{\mathbf{x}}^\top\mathbf{t}(\bar{\mathbf{x}})$. The price of anarchy is defined as the ratio between the overall system cost at Nash equilibrium and the optimum system cost, i.e., $\bar{\mathbf{x}}^\top\mathbf{t}(\bar{\mathbf{x}})/\hat{\mathbf{x}}^\top\mathbf{t}(\hat{\mathbf{x}})$. Many bounds for the noncooperative congestion games have been established, see [1–3,5,6,9], for example.

3. Bounds for the price of anarchy

In this section, we derive several bounds for the price of anarchy for non-atomic congestion games with elastic demand and asymmetric cost functions.

We need the following result to derive our bounds.

Lemma 3.1 *([1, Lemma 4.2]). If every $d_i(n_i)$ is a nonincreasing function for $n_i \geq 0$, then*

$$\sum_{i\in\mathcal{I}} \int_0^{\hat{n}_i} d_i(z)\mathrm{d}z \leq \sum_{i\in\mathcal{I}} \int_0^{\bar{n}_i} d_i(z)\mathrm{d}z + d(\bar{n})^\top(\hat{n}-\bar{n}). \tag{4}$$

Proof. Since d is a nonincreasing function, $\int_0^{\bar{n}_i} d_i(z)\mathrm{d}z$ is a concave function and (4) follows directly. $\square$

To derive the bounds, we have to introduce two quantities for measuring the degree of asymmetry and the nonlinearity. The following definition is from Definition 2 of [2].

Definition 3.1. The quantity c^2 for the nonlinear function $\mathbf{t}$, which measures the asymmetry of the Jacobian $\nabla\mathbf{t}(\mathbf{x})$, is defined as

$$c^2 \equiv \sup_{\mathbf{x}\in X} \|S(\mathbf{x})^{-1}\nabla\mathbf{t}(\mathbf{x})\|^2_{S(\mathbf{x})}, \tag{5}$$

where

$$S(\mathbf{x}) = \frac{\nabla\mathbf{t}(\mathbf{x}) + \nabla\mathbf{t}(\mathbf{x})^\top}{2}$$

is the symmetrized part of the Jacobian matrix $\nabla\mathbf{t}(\mathbf{x})$. $\square$

The following definition was first used in [11,12] in the analysis of interior point methods. It finds new applications recently in the analysis of the price of anarchy [2].

Definition 3.2. The function $\mathbf{t} : R^n \to R^n$ is said to satisfy the **Jacobian similarity** condition if there exists $\kappa \geq 1$ such that $\forall \mathbf{w}\in R^n, \forall \mathbf{x},\ \bar{\mathbf{x}} \in X$, there holds

$$\kappa^{-1}\mathbf{w}^\top\nabla\mathbf{t}(\mathbf{x})\mathbf{w} \leq \mathbf{w}^\top\nabla\mathbf{t}(\bar{\mathbf{x}})\mathbf{w} \leq \kappa\mathbf{w}^\top\nabla\mathbf{t}(\mathbf{x})\mathbf{w}. \quad \square$$

Theorem 3.1. *Suppose that $\mathbf{t}(\mathbf{x})$ satisfies the Jacobian similarity condition and that $\mathbf{t}(\mathbf{0})^\top x \geq 0$ for all $x \in X(\mathbf{n})$, $\mathbf{n}\geq 0$. Then*

$$E(\hat{\mathbf{x}},\hat{\mathbf{n}})/E(\bar{\mathbf{x}},\bar{\mathbf{n}}) \leq \left(1-\frac{c^2\kappa}{4}\right) + \frac{c^2\kappa}{4}\cdot\frac{U(\bar{\mathbf{x}},\bar{\mathbf{n}})}{E(\bar{\mathbf{x}},\bar{\mathbf{n}})}.$$

Proof. Since $(\bar{\mathbf{x}}, \bar{\mathbf{n}})$ is a solution of the variational inequality problem (2) and $(\hat{\mathbf{x}}, \hat{\mathbf{n}})$ is the system optimum solution, we have

$$(\hat{\mathbf{x}} - \bar{\mathbf{x}})^\top \mathbf{t}(\bar{\mathbf{x}}) - \mathbf{d}(\bar{\mathbf{n}})^\top (\hat{\mathbf{n}} - \bar{\mathbf{n}}) \geq 0.$$

Combining this inequality and (4) and rearranging terms, we have

$$\sum_{i \in \mathcal{I}} \int_0^{\bar{n}_i} d_i(\omega)\mathrm{d}\omega - \bar{\mathbf{x}}^\top \mathbf{t}(\bar{\mathbf{x}}) - \left(\sum_{i \in \mathcal{I}} \int_0^{\hat{n}_i} d_i(\omega)\mathrm{d}\omega - \hat{\mathbf{x}}^\top \mathbf{t}(\hat{\mathbf{x}}) \right) + \left(\mathbf{t}(\bar{\mathbf{x}}) - \mathbf{t}(\hat{\mathbf{x}})\right)^\top \hat{\mathbf{x}} \geq 0. \tag{6}$$

From the mean value theorem, there is some $\alpha \in [0, 1]$, such that

$$\mathbf{t}(\bar{\mathbf{x}}) - \mathbf{t}(\hat{\mathbf{x}}) = \nabla \mathbf{t}(\mathbf{x_1})(\bar{\mathbf{x}} - \hat{\mathbf{x}}),$$

where

$$\mathbf{x_1} = \hat{\mathbf{x}} + \alpha(\bar{\mathbf{x}} - \hat{\mathbf{x}}).$$

Thus,

$$\begin{aligned}
(\mathbf{t}(\bar{\mathbf{x}}) - \mathbf{t}(\hat{\mathbf{x}}))^\top \hat{\mathbf{x}} &= \hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})(\bar{\mathbf{x}} - \hat{\mathbf{x}}) \\
&= -\hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\hat{\mathbf{x}} + \bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})^\top S(\mathbf{x_1})^{-1} S(\mathbf{x_1})\hat{\mathbf{x}} \\
&\leq -\hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\hat{\mathbf{x}} + \|\bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})^\top S(\mathbf{x_1})^{-1}\|_{S(\mathbf{x_1})} \|\hat{\mathbf{x}}\|_{S(\mathbf{x_1})} \\
&\leq -\hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\hat{\mathbf{x}} + \|\bar{\mathbf{x}}\|_{S(\mathbf{x_1})} \|S(\mathbf{x_1})^{-1} \nabla \mathbf{t}(\mathbf{x_1})\|_{S(\mathbf{x_1})} \|\hat{\mathbf{x}}\|_{S(\mathbf{x_1})} \\
&\leq -\hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\hat{\mathbf{x}} + c\|\bar{\mathbf{x}}\|_{S(\mathbf{x_1})} \|\hat{\mathbf{x}}\|_{S(\mathbf{x_1})},
\end{aligned} \tag{7}$$

where the first inequality follows from the Cauchy–Schwarz inequality, the second one from the norm inequality, and the last one from (5). For any two vectors $\mathbf{x}$ and $\mathbf{y}$ in R^n, we have

$$2\sqrt{b_1 b_2}\|\mathbf{x}\|_S \|\mathbf{y}\|_S \leq b_1 \|\mathbf{x}\|_S^2 + b_2 \|\mathbf{y}\|_S^2$$

if $b_1, b_2 \geq 0$. This implies that

$$c\|\mathbf{x}\|_S \|\mathbf{y}\|_S \leq b_1 \|\mathbf{x}\|_S^2 + b_2 \|\mathbf{y}\|_S^2 \tag{8}$$

if $b_1, b_2 \geq 0$ and $b_1 b_2 \geq c^2/4$. It follows from (7) and (8) that

$$\begin{aligned}
(\mathbf{t}(\bar{\mathbf{x}}) - \mathbf{t}(\hat{\mathbf{x}}))^\top \hat{\mathbf{x}} &\leq -\hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\hat{\mathbf{x}} + \frac{c^2}{4} \cdot \bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\bar{\mathbf{x}} + \hat{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\hat{\mathbf{x}} \\
&= \frac{c^2}{4} \cdot \bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\bar{\mathbf{x}}.
\end{aligned} \tag{9}$$

Applying again the mean value theorem, we assert that there is $\beta \in [0, 1]$, such that

$$(\mathbf{t}(\bar{\mathbf{x}}) - \mathbf{t}(\mathbf{0}))^\top (\bar{\mathbf{x}} - 0) = \bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_2})\bar{\mathbf{x}}, \quad \text{where } \mathbf{x_2} = \bar{\mathbf{x}} - \beta\bar{\mathbf{x}}. \tag{10}$$

Since $\mathbf{t}(\mathbf{0}) \geq 0, \bar{\mathbf{x}} \geq 0$, it follows that

$$\bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_2})\bar{\mathbf{x}} \leq \mathbf{t}(\bar{\mathbf{x}})^\top \bar{\mathbf{x}}. \tag{11}$$

Then, (7) and (11) and Jacobian similarity mean that

$$(\mathbf{t}(\bar{\mathbf{x}}) - \mathbf{t}(\hat{\mathbf{x}}))^\top \hat{\mathbf{x}} \leq \frac{c^2}{4} \cdot \bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_1})\bar{\mathbf{x}} \leq \frac{c^2}{4}\kappa \cdot \bar{\mathbf{x}}^\top \nabla \mathbf{t}(\mathbf{x_2})\bar{\mathbf{x}} \leq \frac{c^2}{4}\kappa \bar{\mathbf{x}}^\top \mathbf{t}(\bar{\mathbf{x}}). \tag{12}$$

Substituting (12) into (6), we have

$$\sum_{i \in I} \int_0^{\bar{n}_i} d_i(\omega)\mathrm{d}\omega - \left(1 - \frac{c^2\kappa}{4}\right) \bar{\mathbf{x}}^\top \mathbf{t}(\bar{\mathbf{x}}) - \left(\sum_{i \in I} \int_0^{\hat{n}_i} d_i(\omega)\mathrm{d}\omega - \hat{\mathbf{x}}^\top \mathbf{t}(\hat{\mathbf{x}}) \right) \geq 0,$$

i.e.,

$$E(\hat{\mathbf{x}}, \hat{\mathbf{n}}) \leq \left(1 - \frac{c^2\kappa}{4}\right) E(\bar{\mathbf{x}}, \bar{\mathbf{n}}) + \frac{c^2\kappa}{4} U(\bar{\mathbf{x}}, \bar{\mathbf{n}}),$$

and the assertion of the theorem follows immediately. □

If the cost function $\mathbf{t}$ is linear, i.e., $\mathbf{t}(\mathbf{x}) = \mathbf{M}\mathbf{x} + q$, where M is an $F \times F$, positive definite and possibly asymmetric matrix, and $q^\top \mathbf{x} \geq 0$ for all $\mathbf{x} \in X(\mathbf{n})$, then the constant $\kappa = 1$. The degree of asymmetry of a matrix M is

$$c^2 \equiv \|S^{-1}M\|_S^2 \equiv \sup_{w \neq 0} \frac{\|S^{-1}Mw\|_S^2}{\|w\|_S^2} = \sup_{w \neq 0} \frac{w^\top M^\top S^{-1} M w}{w^\top S w},$$

where $S = \frac{M+M^\top}{2}$ is the symmetrized part of the matrix M and $\|w\|_S \equiv \sqrt{w^\top S w}$ denotes the S-norm of a vector w.

It is obvious that when $\mathbf{M}$ is positive definite and symmetric, then, $c^2 = 1$. The constant c^2 was originally introduced by Hammond [13] and has the following property:

Lemma 3.2. *If M^2 is a positive semidefinite matrix, then $c^2 \leq 2$.* □

From Theorem 3.1, we have the following corollary immediately, which provides a bound of the price of anarchy for the noncooperative congestion games with elastic and asymmetric linear cost functions.

Corollary 3.1. *Suppose that $\mathbf{t}(\mathbf{x}) = \mathbf{M}\mathbf{x} + q$, $\mathbf{M}$ is positive definite and $q^\top \mathbf{x} \geq 0$ for all $\mathbf{x} \in X$. Let $\bar{\mathbf{x}}$ be a solution of the user optimization problem (2) and let $\hat{\mathbf{x}}$ be a solution of the system optimization problem (3). Then*

$$E(\hat{\mathbf{x}}, \hat{\mathbf{n}})/E(\bar{\mathbf{x}}, \bar{\mathbf{n}}) \leq \left(1 - \frac{c^2}{4}\right) + \frac{c^2}{4} \cdot \frac{U(\bar{\mathbf{x}}, \bar{\mathbf{n}})}{E(\bar{\mathbf{x}}, \bar{\mathbf{n}})}. \quad \square$$

Remark. For the symmetric case where $\mathbf{M}$ is symmetric positive definite, Chau and Sim [1] derived the bound $3/4 + 1/4 \cdot \frac{U(\bar{v},\bar{n})}{E(\bar{v},\bar{n})}$. Note that for this case, $c^2 = 1$ and our result reduces to Chau and Sim's bound. □

4. Some examples

We now present some examples to illustrate our results.

Example 1. The following simple example is used to show that finding the bound of the price of anarchy for the network equilibrium with elastic demand is not as manageable as in the fixed demand case, even if all the cost functions and the demand functions are linear. This example was also used in [1] for the same purpose, but with errors and were corrected in [14].

Consider a network with one OD pair and one link such that $x = n$ and let the link travel time function be $c(v) = av + b$ and the demand function $d(n) = mn + h$, where $a \geq 0$ and $m < 0$, which ensures the existence of the Nash equilibrium. The Nash equilibrium is $\bar{v} = \bar{n} = \frac{h-b}{a-m}$, the social surplus is

$$E(\bar{v}, \bar{n}) = \frac{(h-b)^2}{a-m} \cdot \left(\frac{0.5m - a}{a-m} + 1\right) = \frac{m(h-b)^2}{2(a-m)^2}$$

and the user benefit is

$$U(\bar{v}, \bar{n}) = 0.5m \left(\frac{h-b}{a-m}\right)^2 + \frac{h(h-b)}{a-m}.$$

The system optimal solution is $\hat{v} = \hat{n} = \frac{h-b}{2a-m}$ and the corresponding social surplus is

$$E(\hat{v}, \hat{n}) = \frac{(h-b)^2}{2(2a-m)}.$$

Thus, the price of anarchy is

$$\rho = \frac{E(\hat{v}, \hat{n})}{E(\bar{v}, \bar{n})} = \frac{(m-a)^2}{m(m-2a)}. \tag{13}$$

In this case, $c = 1$ and our bound is

$$\varrho = \frac{3}{4} + \frac{1}{4} \cdot \frac{U(\hat{v}, \hat{n})}{E(\hat{v}, \hat{n})}.$$

It can be seen from (13) that unlike the bound for the fixed demand case, both the price of anarchy and our bound for the elastic demand case depend on the parameter m in the demand function. If $m \to 0$ (the demand is perfectly elastic), both ρ and ϱ tend to infinity. To see this more clearly, we plot ρ and ϱ as a function of $-m$ for $a = 2$, $b = 0.3$, $h = 5$ (Fig. 1).

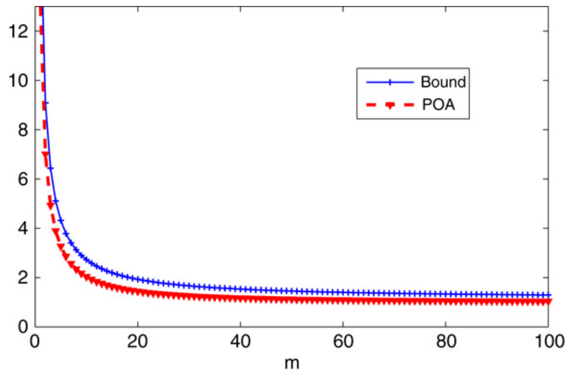

Fig. 1. The price of anarchy and our bound as functions of $-m$.

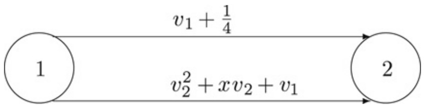

Fig. 2. The Network in Example 2.

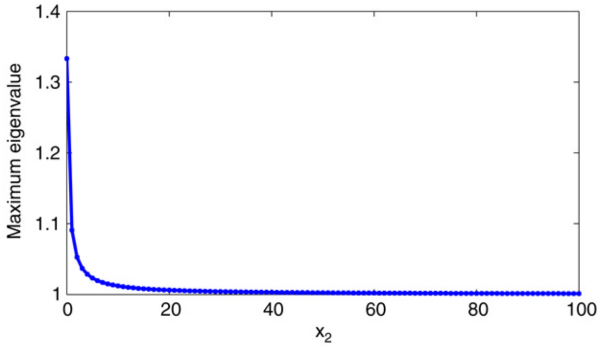

Fig. 3. The maximum of the eigenvalue as a function of x_2.

Example 2. Consider the following network with two nodes and two links (Fig. 2). Suppose that the demand function is $d(n) = -n + 10$, the cost function for link 1 is $v_1 + 1/4$ and the cost function for link 2 is $v_2^2 + v_2 + v_1$, i.e.,

$$c(v) = t(x) = \begin{pmatrix} x_1 + 1/4 \\ x_2^2 + x_2 + x_1 \end{pmatrix}.$$

Then,

$$\nabla t(x) = \begin{pmatrix} 1 & 0 \\ 1 & 2x_2 + 1 \end{pmatrix} \quad \text{and} \quad \kappa = 3.56.$$

From Definition 3.1 we can see that

$$c^2 = \sup_{x \in K} \lambda_{\max}(S(x)^{-1/2} \nabla t(x)^\top S(x)^{-1} \nabla t(x) S(x)^{-1/2}).$$

We thus plot the maximum eigenvalue of matrix $(S(x)^{-1/2} \nabla t(x)^\top S(x)^{-1} \nabla t(x) S(x)^{-1/2})$ for all $x \in [0, 100]$. Fig. 3 indicates that the maximum eigenvalue decreases with respect to x_2 and reaches its maximum at $x_2 = 0$ and $c^2 = 4/3$.

The Nash equilibrium is

$$\bar{v}_1 = \bar{n} = 4.875 \quad \text{and} \quad \bar{v}_2 = 0.$$

The social surplus at Nash equilibrium is

$$E(\bar{v}, \bar{n}) \approx 11.8828$$

and the user benefit is

$$U(\bar{v}, \bar{n}) \approx 36.8672.$$

The system optimum is

$$\hat{v}_1 \approx 1.2748, \qquad \hat{v}_1 \approx 2.4001 \quad \text{and} \quad \hat{n} \approx 3.6749.$$

The optimal social surplus is

$$E(\hat{v}, \hat{n}) \approx 16.8796.$$

Thus, the price of anarchy is

$$\rho = \frac{E(\hat{v}, \hat{n})}{E(\bar{v}, \bar{n})} \approx \frac{16.8796}{11.8828} = 1.4205$$

and our bound is

$$\varrho = \left(1 - \frac{c^2\kappa}{4}\right) + \frac{c^2\kappa}{4} \cdot \frac{U(\bar{\mathbf{x}}, \bar{\mathbf{n}})}{E(\bar{\mathbf{x}}, \bar{\mathbf{n}})} \approx 3.4950.$$

5. Conclusion

In this paper, we proposed some bounds for the price of anarchy of the noncooperative congestion games with elastic demands and asymmetric linear or nonlinear cost functions. The bounds established depend on a constant from the cost functions as well as the ratio between user benefit and social surplus at Nash equilibrium, similar to the bound in [1] for the symmetric case. In this sense, our results can be viewed as generalization of that in [1] from the symmetric case to the asymmetric case.

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